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Signifying the accumulation graph in a dynamic and multi-representation environment

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Abstract The present study focuses on the accumulation process involved in the integration of a single-variable function. Observing the work of two high-school calculus students who had not yet learned any other integral-related ideas, we analyze the emergence of the semiotic relationship between personal and mathematical meanings, as expressed through the understanding of mathematical signs in integration tasks. Adopting Radford's educational perspective whereby learning is defined as a process of objectification, we identify a three-stage evolution of the double semiotic meaning of the lower boundary and of its role in the definition of the accumulation graph: (1) objectifying a zero accumulation in relation to the lower limit, (2) objectifying zero as marking the zero sum of accumulated areas, and (3) objectifying the accumulation graph as dependent on the lower-limit value. This evolution is marked by semiotic changes related to the pivotal role of the "zeros" in the accumulation graph.

Keywords Calculus · Integral · Accumulation function · Objectification · Software artifact · Semiotic mediation

1 Introduction

The concept of integral is considered to be central in the study of calculus, and it is a part of many high-school calculus courses. In many countries, the curriculum attempts to introduce two concepts, the definite or Riemann integral as a computation method and the indefinite integral or the anti-derivative. The fundamental theorem of calculus connects both concepts, linking the derivative with the integral. Meaningful teaching of calculus, and especially of the integral, must consider ways to support the development of these concepts and the awareness of their connection (Thompson, 1994). Technology has been used for the last

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decades mainly at the undergraduate level to support the teaching of calculus. Tools used to teach the concepts of integration include computation tools enhanced by visual representations (Wolfram Alpha, 2010) or Derive (Freese & Stegenga, 2000), and tools designed specifically for the purpose of teaching integrals (Kaskosz, 2004).

Research in the psychology of learning has revealed that the approach to knowledge occurs in two ways: symbolic-reconstructive and perceptual–motor. Calculus, a field involving advanced mathematical concepts, is traditionally taught by transmission of content and procedures presented to the students. In this type of learning, understanding symbols and procedures requires reconstruction of abstract objects, appropriation of symbols, and their mental representation. The cognitive development occurring in such symbolic-reconstructive learning is considered to be a sophisticated way of knowing (Arzarello, Robutti, & Bazzini, 2005). Related to this theoretical framework and in conjunction with development of interactive technology tools, studies in mathematics education emphasize that sensual cognition is essential in the learning process of mathematical concepts (to mention only a few works that are related to the learning of calculus: Arzarello et al., 2005; Bartolini Bussi & Mariotti, 2008; Botzer & Yerushalmy, 2008; Tall, 2009). The case study that we discuss in the present article occurred in settings that support perceptual–motor engagement of students mediated by a computerized artifact and a task designed to lead to the appropriation of particular mathematics content.

Signs in general, and mathematical signs in particular, play two roles. Radford, Bardini, Sabena, Diallo and Simbagoye (2005) define these roles as “social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions” (2005, p.117). Berger (2004), who studied the functional use of mathematical signs, suggests a twofold interpretation of the meaning of signs and objects: personal meaning, “to refer to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not),” and cultural meaning, “to the extent that its usage is congruent with its usage by the mathematical community” (2004, p. 83). In the context of using artifacts, Bartolini Bussi and Mariotti (2008) describe the relations between personal meanings and mathematical meaning as a *double semiotic relationship*. “On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; On the other hand, mathematical meanings may be related to the artifact and its use.” (2008, p. 754). Adopting these terms, we define the double semiotic relationship as the semiotic potential of an artifact, and we assume that the potential is defined with respect to a particular design and pedagogical goals.

The objective of the present study was to analyze the early stages in the development of learning of the accumulation concept by high-school calculus students who had not yet learned any other integral-related ideas, definitions, or procedures. We use the term “stages” in a dynamic sense, suggesting that even though objectification is an ongoing process, it is possible to identify significant points that mark valuable stages. The artifact in our study, shaped by pedagogical considerations and culturally accepted mathematical meaning, is computer software: Calculus UnLimited (CUL) (Schwartz & Yerushalmy, 1996). We considered the elements in CUL (objects such as Cartesian graphs, value table, symbolic icons, etc.) to be the signs that bear meaning accepted in the mathematical culture. Learning in this setting means participating in an active process that leads to making sense of the elements, and bringing about an encounter between personal and mathematical meanings. In other words, to learn something, the learner must become *aware* of its existence within culture (Radford, Bardini, & Sabena, 2007). Becoming aware of an existing mathematical object requires engagement in mathematical activity to grant meaning to the object. Radford (2005) called it “an objectification process.” Objectification requires making use, in a creative way, of different semiotic tools such as words, symbols, and gestures available in the universe of the discourse (Radford, 2003; Radford et al., 2005). Semiotic tools play a central role in the objectification process. Using these tools, students make the transition from the embodied meaning that relates to concrete objects to a disembodied meaning, which is generally related to abstract concepts (Radford, 2005). The presence of a disembodied meaning does not mean that embodiment is removed, for the embodied meaning of previous semiotic activity does not disappear. Embodied meaning “rather gives rise to a more abstract one” (Radford, 2010, p. 45). The design of the present study, the artifact and the tasks, allow us to study the semiotic potential of the artifact and to analyze actions that make the double semiotic relationship related to the accumulation concept observable.

2 Challenges in understanding the concept of the accumulation function

The mental images of the integral concept, the integral as a limit of sums, connections between the various representations of the concept, relations between the definite integral and the computation of area, and understanding the integral as anti-derivative have been the focus of previous research about the challenges of understanding calculus (Machin & Rivero, 2003; Orton, 1983; Rasslan & Tall, 2002; Rösken & Rolka, 2007). These studies have examined the cognitive obstacles encountered while operating in the symbolic world after students learned the concept of integral. This cognitive lens considers thinking as a mental conception and signs as symptom of mental action, thus distinguishing between external (representation of mathematical object) and internal object (indicator of mental actions). The socio-cultural lens considers signs in a comprehensive manner to include words, gestures and actions with artifact. These signs are not an indicator of mental actions; they are considered as an essential component of the thinking process (Radford, 2010).

Few studies identified accumulation as a central element in the understanding of the concept of integral, analogous to the centrality of the concept of slope for the derivative function, and a key concept in understanding the fundamental theorem of calculus (Carlson, Persson, & Smith, 2003; Kouropatov & Dreyfus, 2009; Thompson, 1994). The integral is a complex concept because it can be interpreted in different ways: an anti-derivative process leading to the indefinite integral; the area bounded by a graph and the x-axis; the Riemann sum representing length, area, or volume (this view leading to the definite integral); and the accumulation function, where the upper limit x is a variable, and the lower limit is a fixed parameter. Interpreting the meanings of the various variables participating in the accumulation function is challenging. The accumulation function relates to at least three dynamic elements: the lower limit is a fixed parameter, its value indicating the beginning of the accumulation of area, bounded between the function and the x-axis; the interval Δx, which determines the pace of accumulation and the accuracy of the area computation; and the independent upper limit variable, which determines the value that bounds the accumulation. Each point in the accumulation graph represents the accumulated area (often computed as the product of the dimensions of rectangles) to the x value of this point. Zero values in the accumulation graph represent either the value of the accumulation at the lower limit (A' in Fig. 1) or negative and positive products that accumulated to zero. Understanding the accumulation function requires the awareness that each specific value assigned as a lower limit determines a single accumulation function in a family of accumulation functions. It also requires grasping the meaning of the variables x, $f(x)$, and a,

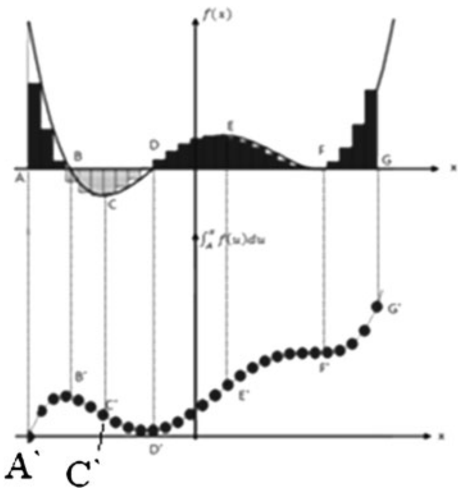


Fig. 1 Accumulation graph

in the symbolic representation $\left(x, f(x), \int_a^x f(u)\mathrm{d}u\right)$, and the dummy variable "u" that appears in a calculation only as a placeholder and disappears completely in the final result. Thompson (1994) found the visual understanding of this simultaneous triple change to be a challenge even for the students who already learned calculus. Thompson (1994), and Thompson and Silverman (2008) studied and articulated two challenges in learning the accumulation function and grasping its idea in relation to other concepts in calculus: understanding that accumulation is a function, and understanding that accumulation occurs at some rate. To grasp the idea of the accumulation function the learner "sees the accumulation and its rate of change as two sides of the same coin" (2008, p. 51). The mutual relationship between the accumulation function and other functions related to calculus concepts, derivative and anti-derivative, could help students see the connection between the concepts of integral and derivative, and conceptualize these links as the fundamental theorem of calculus (Kouropatov & Dreyfus, 2009).

3 Epistemological, semiotic, and pedagogical analysis of the artifact

The interactive integral and accumulation tools at the heart of our study are part of CUL. As a multi-semiotic system, CUL contains different types of signs that we grouped into three categories.

Graphing tools Two Cartesian systems, one above the other, coordinated horizontally. The trajectory in the upper Cartesian system signifies a function. The function is defined symbolically by a free input of a single-variable expression. The trajectory in the bottom system presents the values of $\int_a^x f(u)\mathrm{d}u = g(x) - g(a)$where the derivative of $g(x)$ is the function $f(x)$ or the value of Riemann sums $\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$. The limit of Riemann sums over an interval $[a, b]$ is a real number which may be computed by the definite integral formula $\int_a^b f(x)\mathrm{d}x = g(b) - g(a)$. In another view, it appears as a connected graph and signifies a function whose derivative is the graph in the upper Cartesian system. The associated table of values, displayed on the left, shows the coordinates of the accumulation trajectory.

Accumulation tools Students choose the method of accumulation (right, left, or middle rectangles; trapezoids;  or continuously accumulating area up to graphic and computational limitations (see the icons in Fig. 2)) by clicking an icon or making a menu selection. Because our study focused on the computation of right rectangles, the rectangles that appear on request in the upper Cartesian system represent the product of Δx_i and $f(x_i+\Delta x_i)$. Rectangles are color coded to reflect the product sign (positive or negative). Each dot in the trajectory in the lower Cartesian system is a representation of a Riemann sum (sum of products) expressed mathematically as $\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$.

Boundaries tool After the function f and the method of accumulation are specified, three parameters determine the value of accumulation at a point: the lower and upper boundaries of a bounded region and the width of each division into which the region should be divided. The design of CUL attempts to direct user attention to these parameters and emphasizes especially

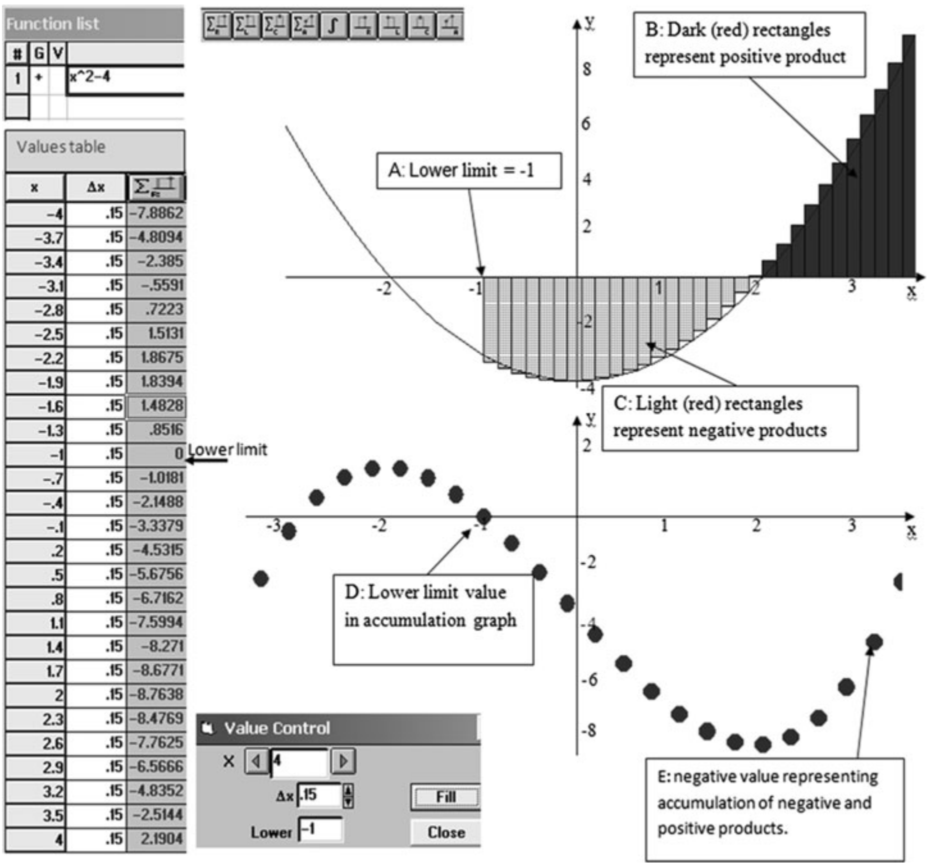


Fig. 2 The CUL integral tools screen

the control over the lower value, with its immediate visual feedback of a value of accumulation of zero. The students control the bounded area by using arrows to move a marker in intervals of Δx to the left or to the right of the lower limit (Δx is an absolute measure of the interval, thus always receiving a positive value both right and left of the lower limit, as shown in Fig. 2). It is also possible to determine the upper value by continuous dragging of the marker. Because our pedagogical decision was to provide direct control of the width (rather than indirect control by specifying the number of splits of a bounded interval), upon determining the lower and upper values, the width of the rectangles is recalculated (as the three variables of boundaries and width of interval are not independent), and the rectangles and the accumulation values are redrawn automatically to have an integer number of rectangles within the specified edges. Another pedagogical consideration related to developing the students' awareness of the effects of the lower value change on the graph of accumulation as a whole.

The graph of the accumulation function is transformed vertically according to interactive changes of the lower limit. This is an interpretation that connects the parameterization of the lower limit with the sign c, representing the indefinite integral as a family of functions that differ by a constant, *c*, or symbolically $\int f(x)\mathrm{d}x = g(x) + c$. The "Fill" tool in the control box represents this decision, as it operates in a mode that computes and draws all points in the bottom window according to the control values.

4 Design of the study

Our case study explored approximately 3 h of learning by Hadel and Shada, two 17-year-old students in the mathematics class taught by Osama Swidan. The experiment took place in school. The students volunteered to participate in three after-school meetings. They already studied the concepts of function and derivative, but not that of integral. The students were familiar with using the derivative symbolically, were able to analyze functions by finding extreme points, and could make link between the properties of a function graph and its related derivative graph. They were familiar with the function graph software that was part of their previous studies of functions, and was used mainly to explore trigonometric functions and to link between functions and derivatives. The two students shared a single computer, and Swidan introduced them briefly to the interface and functionality of the three major tools of the artifact: graphing, accumulating, and controlling. He explained, for example, how to input the function expression, how to operate manually the control buttons, and how to drag the manipulated objects directly. In particular, the students received explanations about the technical functionality and the visual outcome of the various options for representing the area under the curve and of parameters controlled by means of the control box. Because CUL provides a wide repertoire of visual representations of accumulation, we specifically asked the students to start by trying out the accumulation of the right-bound rectangles, where each rectangle is determined by the user's choice of the *size* of Δx and the function *graph value* at $x+\Delta x$. Our request was motivated by our conviction that representations with rectangles, both visual and numeric, are more readable than the continuous area representation that is also available, and that representation with rectangles is better linked to the textbook symbols being taught later.

The following task was presented:

> To work on the given task, you will use the Integral tools of CUL. Your task is to come up with a conjecture and explanation about the mathematical relations between the upper and lower graphs.

To create the upper graph, enter any single-variable function.
To create graphs in the lower graph window, select the right rectangle representation  or other representations located above the graph window.
You can also change any given value in the control box (at the bottom left).
Observe the results of your actions in the lower graph window and in the table of values. Values can also be read directly from the graph windows.
You can work as long as you want, until you feel that you can sketch a graph that will appear in the lower window for a given function graph.

The students were video-recorded, and their computer screens were captured. Swidan was present as an observer and available to provide technical and miscellaneous clarifications. Despite the continuous presence of the mathematics authority person throughout the learning process with the artifact, we made it clear that we would attempt to minimize the intervention. Swidan decided to play an active role only during the last period of the activity, when he had to answer some questions.

5 Data analysis

We used attention and awareness, Radford's (2003) categories of objectification of knowledge, to analyze the evolving processes of personal and mathematical meanings. We identified attention exhibited by embodied sensual involvement with the artifact signs and tools, and analyzed awareness focusing on the processes where the embodied meaning acquires more subtle intangible meaning.

We present here the second round of analysis. The first round consisted of reiterative watching of the videos, concentrating on students' actions with the tools, their repeated gestures, argumentation, and interpretations. It also involved searching for the terms frequently appearing in the transcripts of the videos followed by classifying the transcripts into categories. The findings revealed two noticeable processes: (a) connecting the zero point in the function graph as corresponding to the extreme point in the accumulation graph, which had been influenced, most likely, by the already familiar methods of using derivatives to analyze a given function, and (b) perceiving the lower boundary value in the function graph as corresponding to the zero point in the accumulation graph (also related to zero). For the second round of analysis (presented here), we chose the second process, focusing on controlling the boundaries and its implications, partly because of the larger number of segments involving the term "lower" and referring to the visual representation of this boundary in both graph systems. The lower limit or boundary is assumed to be fundamental to the understanding of "measuring from." This idea was completely new to the students who first encountered it in connection with the given task. For the above reasons, we found this category more compelling, better suited for aligning the personal and mathematical meanings. Therefore, we collated chronologically the utterances and gestures in the discourse that were related to this category. Note that integral-related term "limit" is not part of the artifact terminology, and it was not used by the students. They used the term "lower" instead, as it appears in the artifact, which determines the beginning of the accumulation and is analogous to the term "lower limit" as marking this boundary value. In the course of our analysis, we use the two terms interchangeably, indicating when we do so for the sake of clarity and connection with accepted mathematical meanings.

5.1 Objectifying a relative zero

5.1.1 Attention to the existence of the lower boundary

Students started by graphing the function $f(x)=x$, then set the lower value to zero and used the right rectangles tool to obtain an accumulation function graph (Fig. 3). The designed default drawing of the rectangles was such that the rectangles were drawn to the right of the lower limit while the accumulation graph was plotted for the full visible interval.

A1	Hadel:	Set it to zero [Shada sets 0 as the lower-limit value (Fig. 3)].
A2	Shada:	There is no negative at all [points to the accumulation graph].
A3	Hadel:	Yea, because the area is positive… and negative.
A4	Shada:	Sure, it is because the product of positive and positive is positive and negative times negative is positive.

Attempts to correlate between the top and the bottom graphs began by noticing that all the points of the bottom graph were positive [A2]. Students then focused on the upper window, watching the rectangles and mouse-trace on the x-axis right from the origin, indicating positive, and left from the origin, indicating negative [A3]. They referred to the outer contour of the rectangles, indicating that the two triangles had positive areas as the products of the positive function and the positive x-axis, and that the products of points to the left of the origin by the negative values of the function were positive [A4]. Their tracing gestures on the x-axis and the utterances "positive and positive is positive" and "negative times negative" suggested that they divided the x-axis into two sections, positive and negative. As the choice of the lower value coincided with the origin of the Cartesian system, we were yet to determine the students' notion regarding the role of zero. The next transcript helps clarify this point.

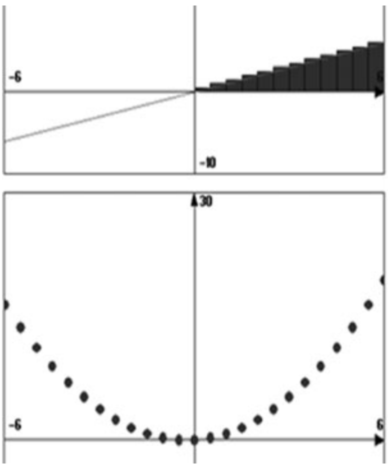


Fig. 3 Lower limit set to 0

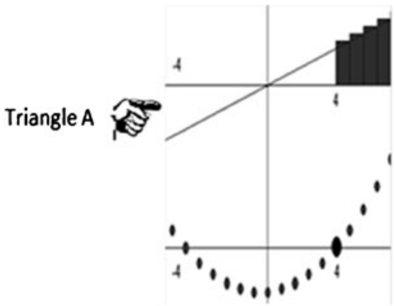
5.1.2 Toward objectification of the lower boundary

The students' next move was to change the lower value from 0 to 4 [B1]. As a result, the area left of the value 4 obtains negative values, and the positive accumulation points graph shows smaller values and shifts downward (Fig. 4). Shada and Hadel immediately noticed this visual change [B2] and attempted to make sense of that view.

B1	Hadel:	Go change the "lower."
B2	Shada:	[Enters 4] What is that? [(Fig. 4) appears on the screen].
B3	Hadel:	I don't know [silence for four seconds]. If I choose that zero [Hadel moves the mouse vertically between the leftmost side of the first rectangle and the corresponding zero] then that is negative [she is mouse-tracing $0<x<4$ in the function graph], then the origin must be negative. Look at that, between four and zero it is negative [points to the bottom window].
B4	Hadel:	There [traces 0 to −4] negative multiplies negatives [triangle A in Fig. 4] is positive [tracing the area left of the origin: triangle A]. But here [tracing the x-axis between the origin and 4] it must be negative because you fixed it [indicates 4 as a lower limit in the upper graph] to zero.
B5	Shada:	No, it is negative between four and negative four. Not between the origin and four.

The terms "negative" and "positive" appear to be central to the conversation, and as students set a lower boundary value different from zero, the term "zero" seems to have acquired a new meaning. As they changed the lower limit to 4, the accumulation graph showed positive and negative values, reflecting the mathematically accepted sign of accumulation of negative and positive area values. The noticeable changes they observed were visually in the [0,4] interval. The presence of negative values in the accumulation graph in the interval challenged the students' previous conjecture about the correlation between the two graphs. Noticing this change caused Hadel to endow the lower-limit value with the property of separation. Hadel's utterance "If I choose that zero" and her gestures (moving with the mouse vertically between the leftmost side of the first rectangle and the corresponding zero), suggest that she noticed that the zero was not the system's origin but rather a separator between the area that was positive to the right and the one that must be negative to the left. "Look at that, between four and zero it is negative" [B3]. She did not consider the negative values of the accumulation graph shown to the left of the origin [B4]. Shada noticed this and rejected Hadel's conclusion [B5]. Indeed, the bounded area to the left of the origin (not filled with rectangles) looks similar to the triangle in the previous episode (Fig. 7) and does not attract visual attention. As the students concluded earlier that it must be positive (negative and negative [A4]), Hadel first expressed her surprise [in B3], then ignored [in B4] the interval $-4<x<0$ where she might have started to detect a contradiction.

Fig. 4 Lower limit set to 4



Overall, at this stage the students considered the non-zero value number on the *x*-axis to separate between positive and negative values. The lower boundary of the drawn area received the accepted mathematical meaning of a zero point in the accumulation function graph that implies positive values to its right and negative values to its left.

The end of this episode is marked by a shift from considering the known origin zero as being the separator of negative and positive values, closer to the mathematical meaning of the lower (limit) value indicating a zero area located anywhere on the *x*-axis of the accumulation graph. We identified this shift from concrete zero to "relative zero" to be the first objectification of mathematical meaning related to accumulation that the students made. Our conviction is based specifically on several utterances and accompanying gestures: "If I choose that zero" and "you fixed it to zero" speaking about an imaginary process that is not a mere description of what they saw in the graph. "If I choose that zero" was accompanied by a gesture indicating the leftmost side of the rectangle, where $x=4$; "you fixed it zero" was accompanied by a gesture pointing to $x=4$ in the upper system. These utterances accompanied by gestures indicate that the students referred to terms that are broader than the concrete objects.

5.2 Objectification of zero as accumulated zero

In the previous episode, attention was paid to a single zero on the bottom graph: the zero that corresponds to the lower-limit value. We conjectured that a more drastic visual change of the area on the right was the reason for the students to focus on that value, ignoring a symmetric zero to the left. This aspect of multiple zeros played a role in the next episode.

5.2.1 Attention paid to the multiple zeros

The students produced a symmetric cubic function $y=x^3$ (Figs. 5 and 6) and set the lower value at (−2).

C1 Hadel: Before this [tracing the *x*-axis for $x<-2$] [(Fig. 5a)] it must be negative and after it [tracing *x*-axis $x>-2$] must be positive, but here it's the opposite! [(Fig. 6)].

C2 Shada: After this [points to *x* and *y* at $x<-2$, Fig. (5b)] it is positive and after that [points to *x* and *y* at $x>2$, (Fig. 5c)] it is positive, but why is it [points to the accumulation] negative between 2 and −2 [Fig. 6]?

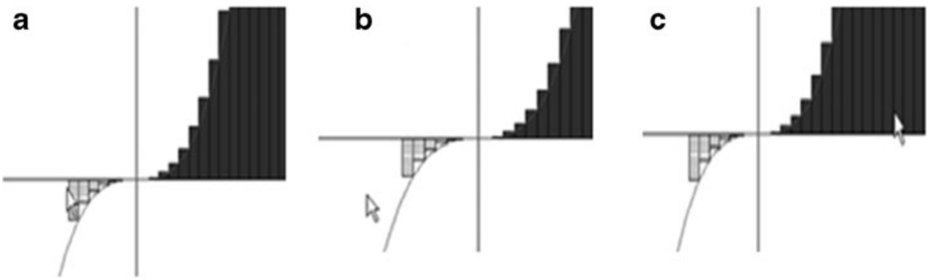
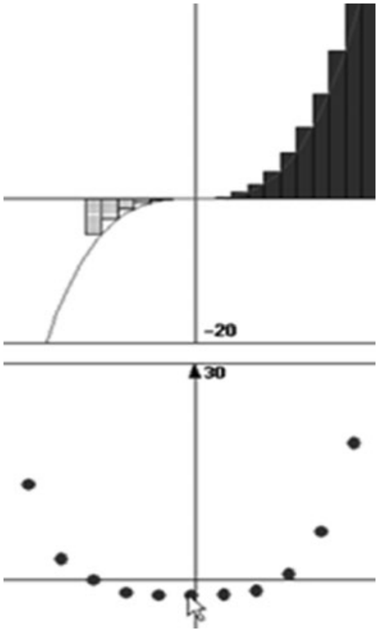


Fig. 5 Lower limit is set to (−2). Students point on the lower limit (**a**), *left* to the lower limit (**b**) and *right* to (+2) at (**c**)

Fig. 6 Lower limit set to −2



According to the “relative zero” notion, Hadel assumed negative values to the left and positive values to the right of the lower boundary value which they fixed at −2 [C1]. The gesture made by Hadel—tracing along the *x*-axis right and left from the lower value, made us wonder about how she correlated between the two graphs. The dominant aspects in the correlation process seemed to be the clear split of the *x*-axis marked by the lower limit of the upper graph. At this stage, the mathematical meaning with which Hadel endowed the artifact sign “lower” as a relative zero having a positive area to its right and a negative one to its left did not allow her to establish the correct mathematical meaning. The mismatch between Hadel’s explanations and the accumulation graph may have motivated Shada to indicate the domains in which the accumulation graph displayed positive values and wonder why this phenomenon occurs [C2]. But while Hadel’s gestures suggested that she was still concerned with changes along the *x*-axis relative to the zero, Shada’s narrative was about domains in the Cartesian system itself, moving toward a two-dimensional observation of area. Observing the accumulation graph, Shada noticed that to the right of the lower boundary value, the accumulation graph obtained negative values all the way to a second zero, and positive values beyond the second zero. This contradicted their previous zero conjecture, and as demonstrated in the next episode, Shada’s gesture probably served as a springboard to move from a local observation of the sign of the rectangle area on the function graph to the accumulation of areas

5.2.2 Awareness of the multiple zeros as accumulated zero

Shifting her observation to the upper Cartesian system, Hadel used the balance metaphor [D1]. She noticed that there were two groups of rectangles of two different colors that seemed to be having identical areas and opposite signs that canceled each other out. She assumed that the area on the left, between −2 and 0, had a value of −2, and that the second area, between 0 and 2, had a value of 2. She claimed that summing these areas would produce a zero [D1]. For the first time, she came closer to the notion of accumulation, shifting the terminology from being about calculation of local shapes, triangles, and rectangles areas (as in the previous episodes) to one about *measuring from left to right* [D1], in this episode. Shada was not convinced by Hadel's explanation [D4]. Hadel provided another numeric example of summation to explain the appearance of the second zero [D5]. This time she pointed to the area between −2 and 0 as −2, and the area from 0 to 1 (which they assumed to be 1.5). Hadel claimed that summing these areas would produce the result −0.5 [D5] and used it to generalize about all the negative values between (−2) and 2. Thus, their shared attention to multiple appearances of zeros, which grew out of the contradiction with the first zero conjecture and the need to explain the second zero, created the awareness of a new aspect of the areas: accumulation.

D1 Hadel: It is maybe the area in 2 and in −2 that cancel each other. When you start to measure from −2, right? The area from 0 to −2 is the same as the area from 0 to 2. As a result the areas cancel each other [(Fig. 7)]. Is that right?

D2 Shada: But why do you get negative?

D3 Hadel: Because the area here [$-2<x<0$] is bigger than the area here [$0<x<1$].

D4 Shada: I am not sure.

D5 Hadel: Look at the area between −2 and 1. The area from −2 and 0 is bigger than the area from 0 to one. That is, the area must be negative. Try to take the area from −2 to 0 as −2 and the area from 0 to 1 as 1 and a half. −2 plus 1.5 is −0.5. That is, we got a negative area. And as a result negative between −2 to 2.

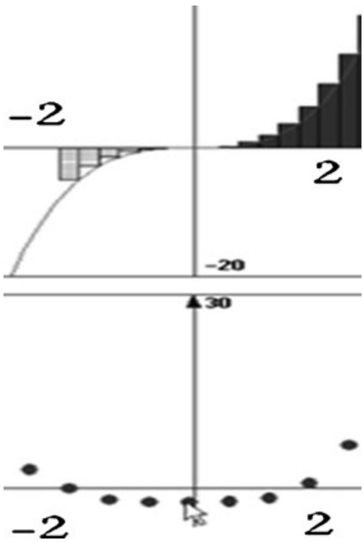


Fig. 7 Areas cancel each other

Thus, becoming aware of the second zero allowed the student to endow it with the mathematical meaning of *zero of accumulation*. This process of noticing and becoming aware of a zero that represents zero accumulation enabled them to endow the negative points in the accumulation graph, which were contradicting previous conjectures, with mathematical meaning: an accumulated value that does not necessarily need to be zero and can also be negative.

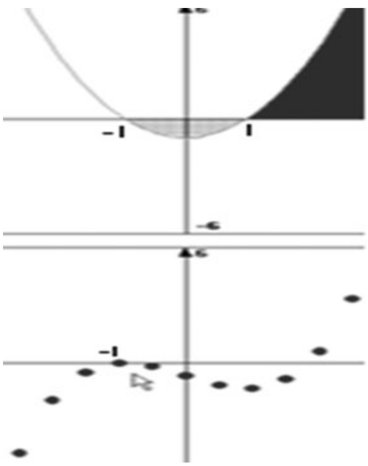
5.3 Objectification of the accumulation

5.3.1 Objectifying the accumulation concept through its graph

In the next episode the students drew a quadratic function and chose to use the continuous area representation instead of the right rectangles they had used so far. They continued to think about the area in terms of dimensions governed by products. In the previous episode, students became aware of the sign of points indicating the accumulation of areas. In the following episode, the focus is on events that demonstrate the deepening of the students' understanding of the objectification of accumulation. In the beginning, the students set the lower boundary to (−1), with the accumulation graph showing zeros at (−1) and at 2 (Fig. 8).

E1	Hadel:	If that [points to −1, Fig. 9a] is zero, then this is positive $x>-1$, (Fig. 9b)] and that is negative [$x<-1$, (Fig. 9c)]. The product of positive by negative is negative. And if there [$x>1$] is positive it [the product] will be positive. That is, between −1 to 1 it [the product] must be negative. How far will it be negative? We need a [positive] area that will be equal to that.
E2	Shada:	But why is this area [between −1 to 0, (Fig. 9d)] negative? It must be positive because the product of negative and negative is positive.
E3	Hadel:	No. When you choose this zero [(Fig. 9a)]. That is positive [indicates to the right of (−1), (Fig. 9b)]. Now this [(Fig. 9a)] as if a zero. As if we shift the *y*-axis to there.
E4	Shada:	That is, this [the area between −1 to 0] is negative and that [the area between 0 to 1] is negative.
E5	Hadel:	[repeating] It means that this is negative and that is negative. We need the area that covers both areas. If we assume that both areas are 2, we should look for an area that equals 2 [pointing to the point and the value (2, 0), (Fig. 9e)]. Good! Here it cancels and after 2 it will become positive [indicates the domain $x>2$ in function graph].

Fig. 8 Lower boundary set to −1



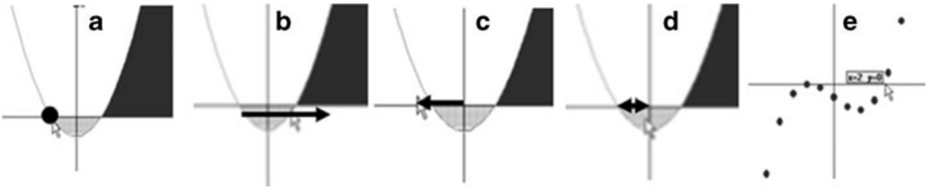


Fig. 9 Lower limit set to −1, students pointing to the area bounded by the function graph at $x=-1$ (**a**), $x>-1$ (**b**), $x<-1$ (**c**), $-1<x<0$ (**d**). Pointing to (2, 0) in the accumulation graph (**e**)

The consolidation of their previous idea about zero representing areas balancing each other appears in E1. Hadel actually implemented it in order to raise a question that she now could answer: "How far will it be negative? We need a [positive] area that will be equal to that." Hadel's attempt to go beyond the explanation based on area balancing reminded Shada of an earlier episode where the y-axis and the first zero coincide. Hadel redivided the x-axis into two parts, positive and negative, as she imagined moving to the y-axis [E3]. Hadel found the area that canceled the area bounded by −1 to 1 when she placed the cursor on the x-axis and showed the y-value as zero at $x=2$. Thus, she concluded that the accumulation area value at that point was zero.

While Shada continued to question the negative and positive areas, Hadel showed elaborated awareness of all previous views on accumulation, mainly by using three types of meaning to signify "positive" and "negative:" (1) "negative" in E1 signifying function position (up or down the x-axis), which was first used in A4; (2) "positive" using relative zeros in E1 and E3 as a sign of splitting the x-axis into domains of negative and positive areas; and (3) the term "negative" indicating a product value as positive or negative. Hadel used that term in her E1 question "until when it will be negative?" indicating a process that had started somewhere and will end once it reaches a zero. At a specific moment, this process receives negative values but goes on to become zero and even continues beyond that, as she articulated in E5 ("Here it cancels and here it will become positive" [indicates the domain $x>2$ in the function graph]). Although the answer to this question does relate to the balance metaphor, the interpretation of the signs and the utterances represents a stronger view of accumulation being a progressive process of summation rather than a binary addition operation, as it was understood earlier.

In the last transcript [E], there are noticeable changes in the students' narrative: the discourse is about processes of accumulation, and their gestures, which earlier marked motion along the x and y axes, shifted to mark the "beginning" of the process. Students did not consider the specific values of points anymore, as they did in episode [D], probably because they already objectified the role of specific zero points in the accumulation graph and endowed it with mathematical meaning.

5.3.2 Objectifying the accumulation through varying lower boundary values

In this episode, Swidan, the interviewer and teacher, took an active role for the first time. Swidan conjectured that students had progressed to a stage where the accumulation graph can be viewed as a function describing phenomena related to the areas bounded between the entire function graph and the x-axis. He wanted to test his conjecture by asking the students directly to vary the lower boundary continuously and test its effect on the functions. Specifically, the next transcript describes the connection between the lower limit and vertical transformation that evolved through the use of the artifact.

F1	Swidan:	What will happen to the accumulation graph when you change the lower-limit value [Swidan uses the term "limit" for lower]?
F2	Hadel:	It will go up and down. [Hadel moves her hand up and down]
F3	Swidan:	Why will it be going up and down?
F4	Hadel+Shada:	It relates to where you measure the area.

Hadel's answer suggests awareness of the fact that changing the lower-limit value relocates the accumulation graph vertically [F2] (Fig. 10). Hadel moves her hand in a way that looks like the vertical transformation of the graph's accumulation that she saw on the computer screen [F2]. The students' answer marks the objectification of the accumulation graph as a sign, having its independent existence and representing positive and negative quantities of areas. To stimulate a further elaboration and explanation, Swidan prompted the students again about the effect of the change of the lower limit on the accumulation graph.

G1	Swidan:	What will happen to the accumulation graph when the lower limit varies? [watching Fig. 11]
G2	Shada:	It will go up and down [moving her hands up and down].
G3	Swidan:	Up and down, why up and down?
G4	Hadel:	When we input zero [for the lower], the area is zero there [(Fig. 11)]. Now, [changes the lower limit to −4, (Fig. 10)] here the area is zero [at −4] and more than zero there [at the origin].
G5	Shada:	At −4 it is zero [the area], to the right of −4 it is positive, more than zero.

Shada noticed that changing the lower limit caused a vertical transformation to take place in the accumulation graph [G2]. Shada's gestures emulated the transformation in the lower boundary of the accumulation graph. Hadel explained this behavior by comparing

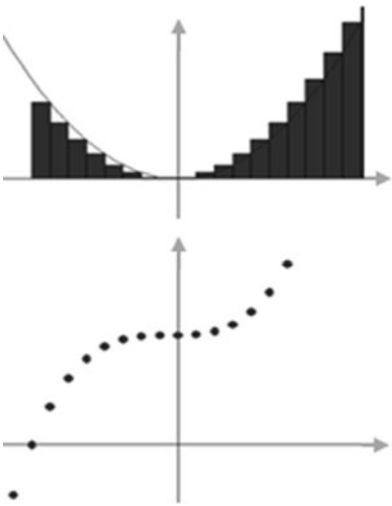
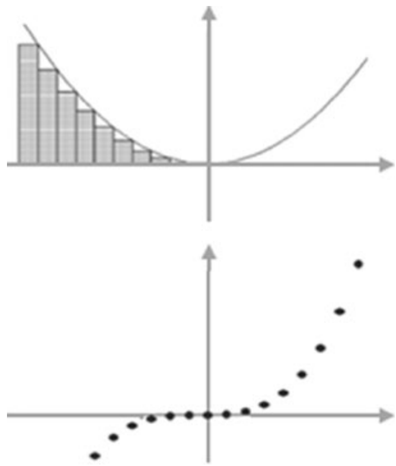


Fig. 10 Lower limit x=−4

Fig. 11 Lower limit $x=0$



area values in two different situations of lower-limit values. Initially, they fixed the lower limit at zero [G4]. Next, they changed the lower limit to −4 and compared the area at −4, which is zero, with the area at the origin, which is greater than zero [G5]. At this stage, the students were no longer dealing with specific point values and moved to explanations that connected between the lower limit, as the initial point of accumulation, and the intersection point with the x-axis, and explained the connection between the area of the rectangles and discrete points in the accumulation graph. The transcript suggests that they referred to the global view of the dynamic changes of the accumulation graph as a result of changing the lower boundary. The gestures of both students, moving their hands up and down, and the use of words like "go up and down," "measure the area," "area is zero," and "area is more than zero" made the vertical transformation apparent to them. In this sense, the accumulation graph has been objectified.

5.4 Summary: the triple objectification process

Shada and Hadel attempted to interpret the connection between the function graph and the dynamically created accumulation point graph which they encountered for the first time while working with the artifact. They experimented first with the given zero lower boundary, attempting to link the positive values of the plotted accumulation points with those of the triangle area bounded between a linear function and the x-axis; they argued that a positive area was the result of the product of a positive x value by a positive function value at that point. At this stage, their conclusions related to a specific case of a linear function for which the calculated area was that of a right triangle, and the zero of the Cartesian system was the zero of the linear function, as well as the value of accumulation at the lower boundary value. After parameterizing the values of the lower boundary, the zero accumulation was no longer the origin of the graph system and led to a new view of the zero value being a relative zero. In reality, there were multiple zero values of accumulation but the students continued to observe a single zero—the first one—on the left that corresponded to the beginning of the drawn rectangles. The personal meaning of the zero which evolved at first, now acquired the mathematical meaning of a zero as any point on the *x*-axis of the accumulation graph marking a split of the axis into two sections, positive and negative. We identified the shift from concrete zero to "relative zero" to be the first objectification of mathematical meaning related to accumulation that the students made. However, this mathematical meaning does not match completely the culturally accepted mathematical meaning of the zero of the accumulation value, as it relates only to cases of negative area to its left and positive area to its right. It was only when the students noted the presence of the multiple zeros in the accumulation graph and revealed positive product values marking what they assumed to be a positive area relating to negative values of the accumulation graph, that their conjecture was revised. Their personal meaning of the notion of the zero was revised when they became aware that zero was not simply splitting between positive and negative values of the area of triangles or rectangles, but represented a point marking the right boundary of an interval upon which the area accumulated to zero. Objectifying the lower boundary as relative zero and objectifying the multiple zero in the accumulation graph as accumulated zero marks the progress from measures of regions or geometric shapes to measuring *from* (the lower boundary) *to* a point (serving as upper boundary) showing awareness of accumulation for the first time. Following a request by the interviewer, Shada and Hadel noticed that a change in the value of the lower bound shifted the accumulation graph vertically. That was not the first time it happened (as shown earlier in Figs. 3 and 4), but they did not notice this change until that later moment. They observed the effect of the shift of the lower boundary as a global change in the accumulation graph that they described as something "transform[ing] upwards" To explain the phenomenon they noticed, the students argued that shifting the lower limit to the left shifts the zero of the accumulation graph left of the original one and subsequently, in the example at hand, the accumulation value increases by a positive value of the rectangle area at that point, therefore the graph as a whole acquires larger values and visually shifts upwards. Their qualitative explanation is indeed the accepted mathematical argument, but an incomplete one, as it attempts to generalize a specific situation. The mathematical argument, however, the terms and the gestures mark another stage of objectification of the accumulation function that concludes this case study.

6 Synthesis and conclusions

In this study, we focused on the accumulation idea involved in the operation of integral of a single-variable function. Analyzing the epistemological structures involved in understanding the concept of integral and drawing on the limited previous research regarding the idea of accumulation, we found that in most cases the obstacles and challenges were studied with students who had already learned the concept of integral in symbolic-reconstructive modes. We took a different point of departure. Our theoretical framework draws on a Vygotskian social–cultural approach that emphasizes the role of attention, awareness, and semiotic mediation. To study the perceptual aspects of this advanced concept of accumulation and to analyze the students' embodied sensuous reasoning mediated by signs and tools, we created a mediating setting, a task and an artifact, that helped us observe high-school calculus students who had not yet learned any integral-related ideas. To investigate the mathematical knowledge that is involved in attaining awareness of sensory information, the designed mediation must manifest both the individual's personal meanings and the constituted cultural system of meanings. To design an experiment that clarified the unstudied aspects involved in learning the accumulation function, we limited to a minimum any direct teaching beyond setting up the mediation. In other words, the study was designed to focus on the task with the artifact, observing its semiotic potential through the analysis of the interplay of personal and cultural meanings The artifact was designed to support exploration through experimentation with dynamic multiple-linked representations of the function and its accumulation function involving interactive changes of parameters, receiving immediate feedback, and allowing direct manipulation of graphic objects. The actions helped us observe the emergence of the double semiotic relationship, the personal meaning and the mathematical meaning, as manifestations of evolutive signs with mediators (Bartolini Bussi & Mariotti, 2008). We observed this development as manifested in the students' use of the artifact as they attempted to understand the complex forms of its signs. To do so, we adopted Radford's (2008) educational perspective of learning as a process of objectification that occurs through active and social engagement with the cultural mathematical knowledge.

Through the semiotic lens, we identified a triple-stage achievement of the double semiotic meaning of the lower boundary and of its role in the definition of the accumulation graph. This evolution is marked by semiotic changes related to the pivotal role of the zeros in the accumulation graph. We found that the occurrence of zero offers interesting information that calls for further attention and explanation. The graphic representation, using color coding to distinguish between negative and positive areas and the symmetric view of a special example of negative and positive areas, created awareness of binary operations between areas—zero as reflecting the balance of negative and positive areas. This balancing metaphor that worked well for special cases later, developed into "summing up from" or accumulating. This development marks the objectification of the zero value as signifying the result of a process of accumulation of negative and positive areas. It promoted the objectification of the non-zero value of the accumulation, negative or positive, as an indication of unbalanced (and therefore non-zero) accumulation. The case study ended with the increased awareness of the accumulation process expressed in a graph that keeps its inherent properties while participating in transformations related to the change of the lower boundary.

The design of the integral artifact was based on the analysis of cognitive studies and on the epistemological analysis of the knowledge related to the integral and accumulation function. The semiotic potential of the design is wide and embraces calculus symbols such as integral and summation, a variety of methods of area accumulation, and it links numeric values and representations with changes of the accumulated area boundaries. Only a narrow part of the potential was explored because we chose the task and the present analysis to focus on the semiotic of the lower boundary of the accumulation that traditional mathematics would term the "lower limit of the definite integral" and limited the experiment by focusing on three meetings. We were surprised that the design, which dynamically provides numeric information related to graphs, points, location, and area of rectangles, did not fully reveal its semiotic potential. We were equally surprised by the overall lack of attention to elements that distinguish between continuous and point graphs and between the area of rectangles "under" the graph and those "covering" the graph, which were assumed to be covering identical areas. The overall ignorance of elements that are usually central in the understanding of accumulation as a computation process stands in contrast with descriptions of challenges of understanding reported about the process of learning the integral by more formal methods (Machin & Rivero, 2003; Orton, 1983; Rasslan & Tall, 2002; Rösken & Rolka, 2007). Clearly, the process described in the present article is an example that should not yet be generalized. The results may be different if mediation is set up to include the paths that we chose to disregard in this experiment, namely teaching intervention and different tasks. Although the role of the teacher as cultural mediator is essential in learning, our goal was to investigate the semiotic potential of the mediators in a yet unexplored domain of semiotic development, and therefore we chose to eliminate additional mediation at this stage. Despite the fact that we limited our investigation to a short case study, without teaching intervention, we were able to observe the objectification of advanced, key concepts in the understanding of integrals, which have not been described before, and which have the potential to shed light on other issues of general interest.

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